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A general Raychaudhuri's equation for second-order differential equations

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Abstract

Raychaudhuri's equation is fundamental for the analysis of behaviour of geodesic congruences. We describe the generalisation to congruences of solutions of arbitrary second-order ordinary differential equations on a manifold. This generalisation allows analysis of the behaviour of congruences generated by specific sets of initial conditions, those invariant under specific Lie group actions as well as singularity analysis. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Our purpose in this paper is to extend the utility of the Raychaudhuri equation from the analysis of geodesic congruences to the study of congruences of solutions of arbitrary second-order ordinary differential equations on manifolds. The practical applications of such a tool are widespread: the focusing of congruences, especially those occurring in constrained dynamics; the analysis of caustics and the global study of singularities of second-order ODEs.

We take as our starting point the study of the Raychaudhuri equation given by Crampin and Prince [1]. In that paper, a tangent bundle approach produces an evolution equation on TM which contains information about all possible geodesic congruences. Pulling this equation back from any given geodesic section and taking the trace gives the Raychaudhuri equation. Tangent bundle techniques have been the basis for the differential geometric study

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of second-order differential equations (see, for example, [2,4,5]) and we have been able to sidestep the apparent obstructions of the absence of metric and linear connection to produce an analogous evolution equation on the evolution space $E := \mathbb{R} \times \mathrm{TM}$.

The structure of the paper is as follows: in Section 2, we review the geodesic Raychaudhuri equation. In Section 3, we present the evolution space formulation of arbitrary, nonautonomous second-order ODEs, in particular the nonlinear connection associated with such equations. In Section 4, we introduce the generalisation, A_Z, of the covariant differential of the vector field Z tangent to a congruence of solution curves and in Section 5, we show that congruence collapse is determined by the zeros of the reciprocal of the trace of this (1,1)-tensor field. Our derivation is fully applicable in the geodesic case and provides an alternative to the usual treatments. Section 6 contains the main result namely the generalisation of the Raychaudhuri equation being the evolution equation for the trace of A_Z. Section 7 lays the foundations for the analysis of congruences produced by sets of initial conditions and gives a version of the Raychaudhuri equation which can be used without an explicit formula for Z. Finally, in Section 8, we analyse planar motion in a magnetic field.

2. Background: the geodesic case

We give a brief account of Crampin and Prince [1]. The setting is an n-dimensional smooth manifold M equipped with a metric g and a symmetric connection ∇ (in fact ∇ need not be the metric connection). We denote the spray of the connection by Γ. Define a type (1,1)-tensor field A_Z associated with a local vector field Z by comparing Lie transport with parallel transport,

$$A_Z := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_t^{-1} \circ \zeta_{t*}. \tag{2.1}$$

Here ζ_t is the flow generated by Z, and τ_t the parallel transport map along ζ_t. There is a simple relationship between A_Z and the covariant derivative. For any ξ tangent to M

$$A_Z(\xi) = \nabla_\xi Z.$$

If Z is geodesic then the propagation equation for A_Z along Z is

$$\mathcal{L}_Z A_Z = \nabla_Z A_Z = -R_Z - A_Z^2, \tag{2.2}$$

where R_Z is the type (1,1)-tensor field obtained from the connection curvature R by

$$R_Z(X) = R(X, Z)Z.$$

Taking the trace of Eq. (2.2) yields Raychaudhuri's equation (taking the trace and Lie differentiation commute)

$$Z(\theta) = -\mathrm{Ric}(Z, Z) - \mathrm{tr}(\omega^2) - \mathrm{tr}(\sigma^2) - \tfrac{1}{n}\theta^2, \tag{2.3}$$

where θ, ω and σ are respectively the divergence, shear and vorticity of Z. That is to say A_Z decomposes into the sum of a multiple of the identity θI, a trace free symmetric part ω

and skew symmetric part σ. Symmetry and skew symmetry are defined with respect to the metric so that ω satisfies

$$g(\omega(X), Y) = g(\omega(Y), X)$$

and σ satisfies the skew symmetric version.

The main result of Crampin and Prince [1] is to show that a global version of Eq. (2.2) on M can be found if one lifts to TM. In this new setting the role of the tensor field A_Z in Eq. (2.2) is played by a type (1,1)-tensor field Q on TM, the vertical projection. Q is a natural geometric object that one has at hand on TM, indeed Q exists by virtue of the fact TM together with ∇ determines a direct sum decomposition of the $2n$-dimensional tangent spaces of TM into n-dimensional vertical and horizontal subspaces. Q is the projection operator onto the vertical subspace.

Let Z be geodesic. The spray of the connection Γ is a global vector field on TM and is an example of a second-order differential equation field. The global version of Eq. (2.2) is

$$Q \circ \mathcal{L}_\Gamma Q = -\hat{R}, \tag{2.4}$$

where $\hat{R}$ is a type (1,1)-tensor field obtained by lifting the connection curvature R on M to TM. Z defines a section σ_Z of the bundle TM $\to M$, such that for any $p \in U \subseteq M$, $\sigma_Z(p) = Z_p$. The section σ_Z is used to pull back so-called vertical tensor fields on TM to M, in particular it is shown that

$$\sigma_Z^* Q = A_Z. \tag{2.5}$$

The significance of this result is that the deformation of the tangent spaces of M by the action of the flow of Z as measured by the definition in Eq. (2.1) is intrinsically available on TM. We exploit this feature in our work in later sections.

When Eq. (2.4) is restricted to the image of the section σ_Z and the restricted equation pulled back to M, Eq. (2.2) is recovered. Moreover, Eq. (2.2) contains information for every geodesic congruence on M, the choice of section determining the congruence. By using the geometric structure of TM endowed with a symmetric connection Crampin and Prince have demonstrated the geometric information of Eq. (2.2) is available in a global form on TM.

Our interest, in this present work, is to generalise these ideas to apply in the case where Z is arbitrary, and hence Γ is no longer geodesic but an arbitrary second-order differential equation field. Our starting point will be a manifold M equipped with a second-order differential equation field Γ. We shall do away with the imposed symmetric connection and take instead the nonlinear connection defined by Γ [2]. In the case where Γ is the geodesic spray, the connection defined by Γ agrees with the metric connection, so the geodesic case outlined above is recoverable as a special case of our generalised framework. In this context, we use Eq. (2.5) as our definition of A_Z, which captures the geometric effect of the action of the flow of Z on M. We derive a propagation equation for A_Z, the trace of which will provide a tool for the singularity analysis of congruences of solution curves, analogous to Raychaudhuri's equation.

3. Evolution space

In Section 2, we worked on TM, which provided a natural setting for the geodesic case. We found that in order to generalise the ideas of that section it was necessary to move to the evolution space $E = \mathbb{R} \times \mathrm{TM}$. We give a brief development, following Crampin et al. [2], of the geometric properties of this new setting.

Let M be a smooth n-dimensional manifold, the configuration space of our system. It will be useful to have the bundle $\pi_0 : \mathbb{R} \times M \to M$. We regard evolution space E as a vector bundle $\pi : \mathbb{R} \times \mathrm{TM} \to \mathbb{R} \times M$ over the graph space $\mathbb{R} \times M$ of M. The fibre $\pi^{-1}(t, x)$ over $(t, x) \in \mathbb{R} \times M$ is the vector space of vectors tangent to M at x. Given any curve $\psi : \mathbb{R} \to M$ such that $\psi(t) = x$ then (t, x, u), where $\dot{\psi}(t) = u$, is an element of $\pi^{-1}(t, x)$. We use (t, x^a, u^a) for $a = 1, \ldots, n$ as adapted coordinates for E. Clearly a curve $\psi : \mathbb{R} \to M$ defines a curve $\mathbb{R} \to E$ by $t \mapsto (t, \psi(t), \dot{\psi}(t))$, the 1-jet of ψ. Such curves are distinguished by the contact 1-forms θ^a, which have the coordinate expression

$$\theta^a := \mathrm{d}x^a - u^a \mathrm{d}t, \tag{3.1}$$

since a curve $\phi : \mathbb{R} \to E$ is the 1-jet of a curve in M if and only if

$$\phi^* \theta^a = 0.$$

Furthermore, the condition

$$\phi^* \mathrm{d}t = \mathrm{d}t$$

ensures that ϕ is parameterised by the time coordinate function t. It follows that any vector field Γ on E whose integral curves are 1-jets of curves in M must satisfy

$$\langle \Gamma, \theta^a \rangle = 0, \quad \langle \Gamma, \mathrm{d}t \rangle = 1.$$

We call such a vector field a second-order differential equation field or SODE. In terms of coordinates,

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + f^a \frac{\partial}{\partial u^a} \tag{3.2}$$

for some smooth local functions f^a on E. Its integral curves have t parameter and satisfy

$$\dot{x}^a(t) = u^a(t), \qquad \dot{u}^a(t) = f^a((t, x(t), u(t))).$$

They are the 1-jets of the solution curves of the second-order differential equations

$$\ddot{x}^a = f^a(t, x, \dot{x}).$$

Let $X \in \mathcal{X}(\mathbb{R} \times M)$, the module of vector fields on $\mathbb{R} \times M$. The prolongation $X^{(1)}$ of X is the unique vector field on E such that

$$\pi_* X^{(1)} = X \quad \text{and} \quad \mathcal{L}_{X^{(1)}} \theta^a \in \mathrm{sp}\{\theta^a\}.$$

In coordinates, given

$$X=\tau\frac{\partial}{\partial t}+\xi^a\frac{\partial}{\partial x^a},\quad \tau,\xi\in C^\infty(\mathbb{R}\times M),$$

then

$$X^{(1)}=\tau\frac{\partial}{\partial t}+\xi^a\frac{\partial}{\partial x^a}+\eta^a\frac{\partial}{\partial u^a},$$

where

$$\eta^a:=\dot{\xi}^a-u^a\dot{\tau}.$$

The fibres of the bundle $\pi:E\to\mathbb{R}\times M$ endow E with a vertical sub-bundle structure. A vector at a point of E is said to be vertical if it is tangent to the fibre of $\pi:E\to\mathbb{R}\times M$. The vector fields $V_a=(\partial/\partial u^a)$ form a local basis of vertical vector fields.

Combining the vertical and contact structure we define a type (1,1)-tensor field S on E by

$$S=V_a\otimes\theta^a. \tag{3.3}$$

S has the following intrinsic properties, which in fact define it:

1. S vanishes on vertical vectors and SODE fields,
2. for any vector field Z on E, $S(Z)$ is vertical,
3. $S(\partial/\partial t)=-\triangle$, where $\triangle=u^a(\partial/\partial u^a)$ (this is a tensorial condition).

In coordinates, given an arbitrary vector field W on E such that $W=\lambda(\partial/\partial t)+\mu^a(\partial/\partial x^a)+\nu^a(\partial/\partial u^a)$, then

$$S(W)=(\mu^a-u^a\lambda)V_a.$$

Let Γ be a SODE. A lot more geometry comes from looking at the deformation of S under the action of Γ. Given the properties of S above it is easy to see

$$\mathcal{L}_\Gamma S(\Gamma)=\mathcal{L}_\Gamma(S(\Gamma))-S(\mathcal{L}_\Gamma\Gamma)=0$$

and

$$\mathcal{L}_\Gamma S(V_a)=\mathcal{L}_\Gamma(S(V_a))-S(\mathcal{L}_\Gamma V_a)=V_a.$$

It is shown in [2] that

$$\mathcal{L}_\Gamma S(H_a)=-H_a,$$

where the n local vector fields H_a are defined relative to Γ (Eq. (3.2)) by

$$H_a:=\frac{\partial}{\partial x^a}-\Gamma_a^b\frac{\partial}{\partial u^a},\quad \text{where}\quad \Gamma_a^b:=-\frac{1}{2}\frac{\partial f^a}{\partial u^b}. \tag{3.4}$$

Thus $\mathcal{L}_\Gamma S$ has eigenvalues $0,+1,-1$. The eigenspace corresponding to the eigenvalue 0 is spanned by Γ, the eigenspace corresponding to the eigenvalue $+1$ is the n-dimensional vertical subspace spanned by the V_a. The n-dimensional subspace corresponding to the eigenvalue -1 is called horizontal, and is spanned by H_a.

The vector fields $\{H_a, V_a, \Gamma\}$ form a local vector field basis on E, with dual basis $\{\theta^a, \psi^a, \mathrm{d}t\}$ where

$$\psi^a := \mathrm{d}u^a - f^a \mathrm{d}t + \Gamma^a_b \theta^b.$$

The Γ^b_a form the components of the nonlinear connection defined by Γ. We remark that Sarlet et al. [5] show that the nonlinear connection arising from a SODE in this way is torsion free.

We define the following type (1,1)-tensor fields: P and Q, projection operators onto the horizontal and vertical subspaces respectively, and N, the projection operator onto the 1-dimensional subspace spanned by Γ. The direct sum decomposition of the tangent spaces of E induced by the eigenspaces of $\mathcal{L}_\Gamma S$ is $I = N + P + Q$ where I is the identity type (1,1)-tensor field. In terms of the dual bases above

$$P = H_a \otimes \theta^a, \qquad Q = V_a \otimes \psi^a, \qquad N = \Gamma \otimes \mathrm{d}t. \tag{3.5}$$

The following equation gives the components of the Jacobi endomorphism $\Phi := Q \circ \mathcal{L}_\Gamma P$, a type (1,1)-tensor field on E:

$$[\Gamma, H_a] = \Gamma^b_a H_b + \Phi^b_a V_b. \tag{3.6}$$

A calculation shows

$$\Phi^b_a = B^b_a - \Gamma^b_c \Gamma^c_a - \Gamma(\Gamma^b_a), \quad \text{where } B^b_a := -\frac{\partial f^b}{\partial x^a}. \tag{3.7}$$

Other useful results

$$[\Gamma, V_a] = -H_a - \Gamma^b_a V_b, \qquad [H_a, H_b] = R^d_{ab} V_d,$$

where R is the curvature of the nonlinear connection Γ^b_a defined by Γ.

In the case, where Γ is the geodesic spray of a symmetric linear connection ∇, the horizontal fields are

$$H_a := \frac{\partial}{\partial x^a} - \Gamma^b_{ac} u^c \frac{\partial}{\partial u^a},$$

the Γ^b_{ac} are the components of ∇, and $\Gamma = u^a H_a$. Therefore, in this case, Φ is related to the curvature by (in terms of components)

$$(\Phi_u)^a_b = R^a_{bcd} u^c u^d, \quad \text{where } [H_a, H_b] = R^d_{cab} u^c V_d.$$

We return to the case where Γ is an arbitrary SODE. Given any vector field $X \in \mathcal{X}(\mathbb{R} \times M)$, we define its vertical lift $X^\vee$ to E by

$$X^\vee = S(X^{(1)}).$$

In coordinates, if $X = \tau(\partial/\partial t) + \xi^a(\partial/\partial x^a)$, then

$$X^\vee = (\xi^a - u^a \tau) V_a.$$

The vertical lift of X depends only on the the base $\mathbb{R} \times M$ since the vertical part of $X^{(1)}$ contributes nothing to $X^{\vee}$ (since $\theta^a(V_b) = 0$). In fact $X^{\vee}$ depends only on the value of X at the point of lifting, that is, the vertical lift works in a pointwise fashion thus we may extend it to vectors tangent to $\mathbb{R} \times M$ by

$$v^{\vee} = \theta^a(v)V_a = (\xi^a - v^a\tau)V_a, \quad \text{where } v = (\tau, \xi)_{(t,x)}.$$

4. The section σ_Z and A_Z

We will now assume the existence of local congruences of (graphs of) solution curves of an arbitrary second-order differential equation (in the geodesic case, this can be established because of the finite separation of conjugate points). The corresponding local tangent vector field is $Z \in \mathcal{X}(\mathbb{R} \times M)$: since the integral curves of Z are graphs, $\mathrm{d}t(Z) = 1$. Hence we write

$$Z = \frac{\partial}{\partial t} + Z^a \frac{\partial}{\partial x^a}, \quad Z^a \text{ local functions on } \mathbb{R} \times M.$$

Z defines a local section σ_Z of $\pi : \mathbb{R} \times \mathrm{TM} \to \mathbb{R} \times M$ as follows:

Definition 4.1. Let $p \in U \subseteq \mathbb{R} \to M$. Then

$$\sigma_Z(p) := (p, \pi_{0*}Z_p).$$

This section will be an important tool for us in what follows. The derivative map $\sigma_{Z*} : \mathbb{R} \times M \to E$ is linear. In coordinates

$$\sigma_Z(t, x^a) = (t, x^a, Z^a)$$

from which the next lemma follows immediately.

Lemma 4.2.

$$\sigma_{Z*}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} + \frac{\partial Z^a}{\partial t}\frac{\partial}{\partial u^a}, \qquad \sigma_{Z*}\left(\frac{\partial}{\partial x^b}\right) = \frac{\partial}{\partial x^b} + \frac{\partial Z^a}{\partial x^b}\frac{\partial}{\partial u^a}.$$

Definition 4.3. We use an overline to indicate the restriction to the image of the section, $\mathrm{Im}(\sigma_Z)$. For example, the restriction of the contact 1-forms is denoted

$$\bar{\theta}^a := \theta^a|_{\sigma_Z(\mathbb{R}\times M)} = \mathrm{d}x^a - Z^a \mathrm{d}t.$$

Note the Z^a are local functions on $\mathbb{R} \times M$. The restriction to the section of the θ^a have the same coordinate formulae as the pullback by the section. We make no notational distinction between the two.

Lemma 4.4.
(i) *Let f^a be as in Eq. (3.2), then $\bar{f}^a = Z(Z^a)$.*
(ii) $\bar{\Gamma} = \sigma_{Z*}(Z)$, *i.e. $\bar{\Gamma}$ is tangent to* $\mathrm{Im}(\sigma_Z)$.

Proof. (i) Let $t \mapsto (t, \phi(t))$ be an integral curve of Z. Restrict f^a to points $(t, \phi(t), \dot{\phi}(t)) \in \mathrm{Im}(\sigma_Z)$. By definition

$$\bar{f}^a = \frac{\mathrm{d}}{\mathrm{d}t} u^a(t, \phi(t), \dot{\phi}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} \dot{\phi}^a(t) = \frac{\mathrm{d}}{\mathrm{d}t} Z^a_{\phi(t)} = Z(Z^a).$$

(ii) This follows from (i) and Lemma 4.2. □

Let F be a local function on E, that is $F = F(t, x^a, u^a)$. Then $\bar{F} = F(t, x^a, Z^a(t, x^b))$. Regarding $\bar{F}$ as a function on the base, the derivatives of $\bar{F}$ in the coordinate directions are

$$\frac{\partial}{\partial t}(\bar{F}) = \overline{\frac{\partial F}{\partial t}} + \frac{\partial Z^a}{\partial t} \overline{\frac{\partial F}{\partial u^a}}, \qquad \frac{\partial}{\partial x^b}(\bar{F}) = \overline{\frac{\partial F}{\partial x^b}} + \frac{\partial Z^a}{\partial x^b} \overline{\frac{\partial F}{\partial u^a}}. \tag{4.1}$$

Definition 4.5. Let B be a type (1,1)-tensor field on E. We say B is vertical if $Q \circ B = B$, that is, the image of B is purely vertical.

We can now define a pullback of vertical type (1,1) tensors on E to $\mathbb{R} \times M$.

Definition 4.6. Let B be a vertical tensor field on E, ie. $Q \circ B = B$. We define the pullback $\sigma_Z^* B$ of B from E to $\mathbb{R} \times M$ in the following way. Given any vector ξ tangent to $\mathbb{R} \times M$ at p, $B(\sigma_{Z*}\xi)$ is vertical, and hence there is a unique vector $\eta \in T_p(\mathbb{R} \times M)$ such that $\mathrm{d}t(\eta) = 0$ and $\eta^\vee = B(\sigma_{Z*}\xi)$. Evidently η depends linearly on ξ and we denote the linear map $T_p(\mathrm{R} \times M) \to T_p(\mathrm{R} \times M)$ which takes ξ to η by $\sigma_Z^* B$. Hence

$$(\sigma_Z^* B(\xi))^\vee = B(\sigma_{Z*}\xi) \quad \text{and} \quad \mathrm{d}t(\sigma_Z^* B) = 0. \tag{4.2}$$

Now we turn Eq. (2.5) into a definition

Definition 4.7. We define the type (1,1)-tensor field A_Z on $\mathbb{R} \times M$ associated with Z by

$$A_Z := \sigma_Z^* Q. \tag{4.3}$$

Proposition 4.8. *The co-ordinate expression for A_Z.*

$$A_Z = \left(\frac{\partial Z^a}{\partial x^b} + \bar{\Gamma}^a_b \right) \frac{\partial}{\partial x^a} \otimes \bar{\theta}^b.$$

Proof. Let $\xi \in T_p(\mathbb{R} \times M)$, in coordinates $\xi = \tau(\partial/\partial t) + \xi^a(\partial/\partial x^a)$. Now $A_Z(\xi)^\vee = Q(\sigma_{Z*}\xi) = \psi^a(\xi + \xi(Z^b)V_b)V_a$. Expanding this yields

$$Q(\sigma_{Z*}\xi) = (-f^a \mathrm{d}t(\xi) + \Gamma^a_b \theta^b(\xi) + \xi(Z^a))V_a$$

and restriction to the section (using Lemma 4.4) gives

$$Q(\sigma_{Z*}\xi) = (-Z(Z^a)\mathrm{d}t(\xi) + \bar{\Gamma}^a_b\bar{\theta}^b(\xi) + \xi(Z^a))V_a$$
$$= \left(-\tau Z^b\frac{\partial Z^a}{\partial x^b} + \xi^b\frac{\partial Z^a}{\partial x^b} + \bar{\Gamma}^a_b\bar{\theta}^b(\xi)\right)V_a = \left(\frac{\partial Z^a}{\partial X^b}\bar{\theta}^b(\xi) + \bar{\Gamma}^a_b\bar{\theta}^b(\xi)\right)V_a$$
$$= \bar{\theta}^b(\xi)\left(\frac{\partial Z^a}{\partial X^b} + \bar{\Gamma}^a_b\right)V_a.$$

Therefore (since $\mathrm{d}t(A_Z(\xi)) = 0$),

$$A_Z(\xi) = (\bar{\theta}^b(\xi))\left(\frac{\partial Z^a}{\partial x^b} + \bar{\Gamma}^a_b\right)\frac{\partial}{\partial x^a}$$

and hence

$$A_Z = \left(\frac{\partial Z^a}{\partial x^b} + \bar{\Gamma}^a_b\right)\frac{\partial}{\partial x^a}\otimes\bar{\theta}^b$$

as required. □

Note that $\mathrm{tr}(A_Z) = (\partial Z^a/\partial x^a) + \bar{\Gamma}^a_a$.

Definition 4.9. We define a covariant derivative-like operator $\bar{\nabla}$ which acts only along Z to be the linear operator with the properties
(i) $\bar{\nabla}(f) := Z(f)$, for all $f \in C^\infty(\mathbb{R}\times M)$,
(ii) $\bar{\nabla}(X) := [Z, X] + A_Z(X)$, for all $X \in \mathcal{X}(\mathbb{R}\times M)$.

This is modelled on the conventional identity $\nabla_Z X - \nabla_X Z = [Z, X]$. Note that this produces an effectively torsion free connection.

Lemma 4.10. *Properties of* $\bar{\nabla}$. *Let* $Y, X \in \mathcal{X}(\mathbb{R}\times M)$ *and* $f \in C^\infty(\mathbb{R}\times M)$. *Then*
(i) $\bar{\nabla}(X+Y) = \bar{\nabla}X + \bar{\nabla}Y$,
(ii) $\bar{\nabla}Z = 0$,
(iii) $\bar{\nabla}(fX) = Z(f)X + f\bar{\nabla}X$,
(iv) *Given* $X = \tau(\partial/\partial t) + X^a(\partial/\partial x^a)$, *then* $\bar{\nabla}X = Z(\tau)(\partial/\partial t) + (Z(X^a) + \bar{\Gamma}^a_b X^b - \tau(Z(Z^a) + \bar{\Gamma}^a_b Z^b))(\partial/\partial x^a)$,
(v) $\bar{\nabla}(\partial/\partial x^a) = \bar{\Gamma}^b_a(\partial/\partial x^b)$,
(vi) $\bar{\nabla}(\partial/\partial t) = -(Z(Z^a) + Z^b\bar{\Gamma}^a_b)(\partial/\partial x^a)$.

Proof. (i) is immediate from the definition. (ii) $\bar{\nabla}Z = [Z, Z] - A_Z(Z) = 0$ since $(A_Z(Z))^\vee = Q(\sigma_{Z*}Z) = Q(\Gamma) = 0$. (iii) follows from the definition. (iv) is a straight forward coordinate calculation. (v) follows from (iv) with $X = (\partial/\partial x^a)$. (vi) follows from (iv) with $X = (\partial/\partial t)$. □

We extend $\bar{\nabla}$ to act on 1-forms by duality. Let $\omega \in \mathcal{X}^*(\mathbb{R}\times M)$ and $X \in \mathcal{X}(\mathbb{R}\times M)$. Then

$$\bar{\nabla}(\omega(X)) = (\bar{\nabla}\omega)(X) + \omega(\bar{\nabla}X). \tag{4.4}$$

Lemma 4.11.
(i) $\bar{\nabla}\mathrm{d}t = 0$,
(ii) $\bar{\nabla}\bar{\theta}^a = -\bar{\Gamma}^a_b\bar{\theta}^b$.

Proof. Use the dual bases $\{Z, (\partial/\partial x^a)\}$ and $\{\mathrm{d}t, \bar{\theta}^a\}$. Using Eq. (4.4) one finds $\bar{\nabla}\mathrm{d}t(Z) = 0$ and $\bar{\nabla}\mathrm{d}t(\partial/\partial x^a) = 0$ which gives (i). While $\bar{\nabla}\bar{\theta}^a(Z) = 0$ and $\bar{\nabla}\bar{\theta}^a(\partial/\partial x^b) = -\bar{\Gamma}^a_b$, gives (ii). □

5. Singularity analysis

In order to study the collapse of the congruence corresponding to the vector field Z we define a volume form $\tilde{\Omega}$ which measures the transverse volume of the congruence. The natural transverse volume form is the characterising form for the distribution sp$\{Z\}$

$$\Omega := \bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n.$$

In the presence of a metric connection, say ∇ from Section 2, the parallel transport τ_t is isometric (length and angle preserving) and one may distinguish [7] a preferred volume form Υ, which in a right handed coordinate system is

$$\Upsilon = |\det(g_{ab})|^{1/2}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

g_{ab} being the components of the metric. Υ has the property that $\nabla\Upsilon = 0$. We use this metric connection picture to motivate a normalisation condition for the transverse volume form Ω, thus, we require our "normalised volume" $\tilde{\Omega}$ to be invariant along the congruence under "parallel transport", that is, $\bar{\nabla}\tilde{\Omega} = 0$.

Proposition 5.1. *Let μ be any smooth function satisfying*

$$Z(\mu) + \mu\frac{\partial Z^a}{\partial x^a} = 0 \tag{5.1}$$

along the congruence defining Z. Then

$$\bar{\nabla}(\mu\Omega) = 0.$$

Proof.

$$\begin{aligned}\bar{\nabla}(\mu\Omega) &= Z(\mu)\Omega + \mu\bar{\nabla}\Omega = Z(\mu)\Omega + \mu(\bar{\nabla}\bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n + \cdots + \bar{\theta}^1 \wedge \cdots \wedge \bar{\nabla}\bar{\theta}^n)\\ &= \left(Z(\mu) + \mu\frac{\partial Z^a}{\partial x^a}\right)\Omega = 0.\end{aligned}$$ □

Let $\tilde{\Omega} := \mu\Omega$. Note that since

$$Z\rfloor\mathcal{L}_Z\tilde{\Omega} = \mathcal{L}_Z(Z\rfloor\tilde{\Omega}) + \tilde{\Omega}(\mathcal{L}_Z Z) = 0,$$

$\mathcal{L}_Z\tilde{\Omega}$ is characterising for sp$\{Z\}$. We now derive the evolution of $\tilde{\Omega}$ under Lie transport along the congruence.

Proposition 5.2.

$$\mathcal{L}_Z\tilde{\Omega} = \theta\tilde{\Omega},$$

where $\theta := \mathrm{tr}(A_Z) = (\partial Z^a/\partial x^a) + \bar{\Gamma}^a_a$.

Proof. The same idea as Proposition 5.1. □

Next, we show that congruence collapse is determined by the zeros of θ^{-1}. In order to have a standardised numerical transverse volume along the congruence we use n Lie-dragged, linearly independent transverse vector fields $X_1, \dots, X_n$ along an integral curve γ of Z, that is $\mathcal{L}_Z X_i = 0$ (the local existence of these fields is guaranteed). Then the numerical transverse volume at $\gamma(s)$ is

$$\tilde{\Omega}(s) := \tilde{\Omega}(X_1(s), \dots, X_n(s))$$

and we have the simple corollary to Proposition 5.2.

Corollary 5.3.

$$\mathcal{L}_Z(\tilde{\Omega}(s)) = \theta(s)\tilde{\Omega}(s), \tag{5.2}$$

where $\theta(s) := \theta(\gamma(s))$.

The differential equation (5.2) has formal solution

$$\tilde{\Omega}(s) = \tilde{\Omega}(0)\exp\left(\int_0^s \theta(s)\mathrm{d}s\right) \tag{5.3}$$

on an interval I on which θ is continuous. Note that if the integral $\int_0^s \theta(s)\mathrm{d}s$ diverges to $-\infty$ then the numerical volume $\tilde{\Omega}(s)$ collapses. Suppose $I = [0, a)$ and $(1/\theta)$ has no zeros on I but has a zero at a. Then it is simple to show a sufficient condition for the divergence of the integral is that $\theta < 0$ and $(1/\theta)' > 0$ on I and the derivative of $(1/\theta)$ exists at a. Raychaudhuri's equation is used to determine the behaviour of the function $\theta := \mathrm{tr}(A_Z)$, on which the singularity analysis of the congruence depends.

6. The evolution of A_Z

In this section, we exhibit an evolution equation for A_Z, which in turn gives an evolution equation for $\mathrm{tr}(A_Z)$. The next lemma allows us to simplify the proof of the theorem.

Lemma 6.1.
(i) $-\bar{B}^a_b = Z(\partial Z^a/\partial x^b) + \partial Z^c/\partial x^b((\partial Z^a/\partial x^c) + 2\bar{\Gamma}^a_c)$,
(ii) $Z(\bar{\Gamma}^a_b) = \overline{\Gamma(\Gamma^a_b)}$.

Proof.
(i) $-\bar{B}^a_b = (\overline{\partial f^a}/\partial x^b) = (\partial \bar{f}^a/\partial x^b) - (\partial Z^c/\partial x^b)(\overline{\partial f^a}/\partial u^c)$, where we have used Eq. (4.1). Using Lemma 4.4 we have $-\bar{B}^a_b = (\partial/\partial x^b)(Z(Z^a)) + 2(\partial Z^c/\partial x^b)\bar{\Gamma}^a_c$, expanding the first term on the RHS gives the required result.

(ii) One has

$$Z(\bar{\Gamma}^a_b) = \frac{\partial}{\partial t}(\bar{\Gamma}^a_b) + Z^c \frac{\partial}{\partial x^c}(\bar{\Gamma}^a_b) = \overline{\frac{\partial \Gamma^a_b}{\partial t}} + \bar{u}^c \overline{\frac{\partial \Gamma^a_b}{\partial x^c}} + \bar{f}^c \overline{\frac{\partial \Gamma^a_b}{\partial u^c}} = \overline{\Gamma(\Gamma^a_b)},$$

where, once again, we have used Eq. (4.1). □

We now come to the main theorem.

Theorem 6.2.

$$\mathcal{L}_Z A_Z = -A_Z^2 - \bar{\Phi}.$$

Proof. Let $D^a_b := (\partial Z^a/\partial x^b) + \bar{\Gamma}^a_b$. Then $A_Z = D^a_b(\partial/\partial x^a) \otimes \bar{\theta}^b$ and clearly $A_Z^2 = D^a_c D^c_b(\partial/\partial x^a) \otimes \bar{\theta}^b$. We now approach the proof head on.

$$\mathcal{L}_Z A_Z + A_Z^2 = \left(Z(D^a_b) - D^c_b \frac{\partial Z^a}{\partial x^c} + D^a_c \frac{\partial Z^c}{\partial x^b} + D^a_c D^c_b \right) \frac{\partial}{\partial x^a} \otimes \bar{\theta}^b.$$

Expanding the RHS, cancelling terms and using Lemma 6.1(ii) gives

$$\left(\frac{\partial}{\partial x^b}(Z(Z^a)) + 2\bar{\Gamma}^a_c \frac{\partial Z^c}{\partial x^b} + \overline{\Gamma(\Gamma^a_b)} + \bar{\Gamma}^a_c \bar{\Gamma}^c_b \right) \frac{\partial}{\partial x^a} \otimes \bar{\theta}^b.$$

Expanding the first term and rearranging yeilds

$$\left(Z\left(\frac{\partial Z^a}{\partial x^b}\right) + \frac{\partial Z^c}{\partial x^b}\left(\frac{\partial Z^a}{\partial x^c} + 2\bar{\Gamma}^a_c\right) + \overline{\Gamma(\Gamma^a_b)} + \bar{\Gamma}^a_c \bar{\Gamma}^c_b \right) \frac{\partial}{\partial x^a} \otimes \bar{\theta}^b.$$

One may now recognise $-\overline{B}^a_b$ from Lemma 6.1(i), and so the expression becomes

$$(-\overline{B}^a_b + \overline{\Gamma(\Gamma^a_b)} + \bar{\Gamma}^a_c \bar{\Gamma}^c_b)\frac{\partial}{\partial x^a} \otimes \bar{\theta}^b = -\overline{\Phi}$$

as required. □

We conclude this section with a result which shows that Theorem 6.2 is the pullback of a global evolution equation (cf. the goedesic case described in Section 2).

Proposition 6.3.

$$Q \circ \mathcal{L}_\Gamma Q = -\Phi.$$

Proof. Recall $I = N + P + Q$, where N is the projection onto $\mathrm{sp}\{\Gamma\}$, the eigenspace of $\mathcal{L}_\Gamma S$ corresponding to eigenvalue 0. It is straightforward to check that $\mathcal{L}_\Gamma N = 0$. Hence,

$$Q \circ \mathcal{L}_\Gamma Q = Q \circ \mathcal{L}_\Gamma (I - N - P) = -Q \circ \mathcal{L}_\Gamma P = -\Phi. \qquad \square$$

It follows as an immediate corollary to Proposition 6.3 that $\mathcal{L}_Z A_Z + A_Z^2 = -\bar{\Phi}$ (Theorem 6.2) is the pullback of $Q \circ \mathcal{L}_\Gamma Q = -\Phi$.

7. Computational issues

In order to analyse A_Z and the generalised Raychaudhuri equation in concrete cases where an explicit form for a local vector field Z is not known we derive an alternative expression for A_Z in which the derivatives of the components of Z do not appear. This expression is particularly useful in dealing with SODEs possessing symmetry. In what follows we will *construct* n commuting vector fields which commute with our SODE Γ. In practical examples (see Section 8) we would expect to use symmetries of the SODE with these properties. In effect, the presence of such symmetries produces local congruences of solution curves on $\mathbb{R} \times M$ along which particular sets of first integrals have constant values; we do not, however, require expressions for these integrals in our alternative definition of A_Z.

Let S^0 be the image of a section of $\pi : \{0\} \times \mathrm{TM} \to \{0\} \times M$ so that it is an n-dimensional submanifold of $\{0\} \times \mathrm{TM}$ with nowhere vertical tangent space and with Γ nowhere tangent to S^0. We make this requirement so that there is exactly one projected solution curve passing through each point on the projection of S^0. Let $\{X_a^0\}$ be n commuting tangent vector fields to S^0 and define

$$X_a(t) := \zeta_{t*}(X_a^0)$$

for all $t \in I := [0, T]$ for which $\mathrm{sp}\{X_a(t)\}$ contains no vertical direction and for which $\dim(\mathrm{sp}\{X_a(t)\}) = n$ ($T > 0$ by continuity). Now $[X_a(t), X_b(t)] = 0$, $[X_a(t), \Gamma] = 0$ and $\mathrm{sp}\{X_a(t)\}$ is the tangent distribution to the n-dimensional submanifold $S^t := \zeta_t^* S^0$. Let $S := \cup_{t \in I} S^t$. Notice that Γ is tangent to this $(n+1)$-dimensional submanifold of E and that the restriction of π to S is one-to-one and onto its image. As a result we define a vector field on $\pi(S)$ by

$$Z_{\pi(p)} := \pi_* \Gamma_p$$

for each $p \in S$ and hence a local section of $\pi : E \to \mathbb{R} \times M$ as in Definition 4.1 so that $\sigma_Z(\pi(S)) = S$, $\sigma_{Z*}(Z) = \bar{\Gamma}$ (using our previous notation) and

$$[Z, \pi_* X_a]_{\pi(p)} = 0 = [\pi_* X_a, \pi_* X_b]_{\pi(p)}$$

for each $p \in S$, using the facts that $[X_a, X_b] = 0$ and

$$[\pi_* \Gamma, \pi_* X_a]_{\pi(p)} = \pi_*([\Gamma, X_a]_p) = 0.$$

We will use this section to obtain a simple expression for A_Z which contains no derivatives of Z. Let

$$\sigma_{Z_*}\left(\frac{\partial}{\partial t}\right) = E_0^0 \bar{\Gamma} + E_0^b \bar{X}_b$$

and

$$\sigma_{Z_*}\left(\frac{\partial}{\partial x^a}\right) = E_a^0 \bar{\Gamma} + E_a^b \bar{X}_b.$$

Lemma 7.1.

$$E_0^0 = 1, \quad E_0^b \bar{X}_b^c = -Z^c, \quad E_0^b \overline{\Gamma(X_b^c)} + \bar{f}^c = \frac{\partial Z^c}{\partial t},$$
$$E_a^0 = 0, \quad E_a^b \bar{X}_b^c = \delta_a^c, \quad \frac{\partial Z^b}{\partial x^a} = E_a^c \overline{\Gamma(X_c^b)}.$$

Proof. These results follow from Lemma 4.2 and the observation that $[X_a, \Gamma] = 0$ implies (but is not equivalent to)

$$X_a = X_a^b \frac{\partial}{\partial x^b} + \Gamma(X_a^b) \frac{\partial}{\partial u^b} = X_a^b H_b + (\Gamma_d^b X_a^d + \Gamma(X_a^b)) V_b. \qquad \square$$

Theorem 7.2.

$$A_Z = (\bar{\Gamma}_b^a + E_b^c \overline{\Gamma(X_c^a)}) \bar{\theta}^c \otimes \frac{\partial}{\partial x^a}$$

and

$$\theta := \mathrm{tr}(A_Z) = \bar{\Gamma}_a^a + E_a^c \overline{\Gamma(X_c^a)}.$$

Proof. The proof follows from Proposition 4.8 with the use of Lemma 7.1. $\square$

In the next section, we demonstrate how the formulae of Theorem 7.2 can be used when we have explicit symmetries satisfying the defining relations for the X_a.

8. Example

A charged particle moving in a plane under the influence of a constant vertical magnetic field $\underline{B}$ satisfies

$$\frac{\mathrm{d}\underline{v}}{\mathrm{d}t} = q \underline{v} \times \underline{B}. \tag{8.1}$$

The corresponding SODE is

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + Bu^2 \frac{\partial}{\partial u^1} - Bu^1 \frac{\partial}{\partial u^2}.$$

It is a simple matter to show that

$$[\Phi_b^a] = \begin{pmatrix} \frac{1}{4}B^2 & 0 \\ 0 & \frac{1}{4}B^2 \end{pmatrix} \quad \text{and} \quad \mathrm{tr}(\Phi) = \tfrac{1}{2}B^2.$$

The equations of motion have four obvious symmetries, generated by the following commuting fields on E:

$$Y_1 := \frac{\partial}{\partial t}, \quad Y_2 := \frac{\partial}{\partial x}, \quad Y_3 := \frac{\partial}{\partial y}, \qquad Y_4 := -u^2 \frac{\partial}{\partial x^1} + u^1 \frac{\partial}{\partial x^2} + Bu^a \frac{\partial}{\partial u^a},$$

all of which commute with Γ. In order to have fields tangent to constant time slices of $\mathbb{R} \times \mathrm{TM}$ we will use $X_1 := Y_1 - \Gamma$, and $X_\alpha := Y_\alpha$, $\alpha = 2, 3, 4$. Taking these fields in pairs we could, in theory, construct six 2-parameter families of local vector fields Z tangent to local congruences of solution curves of the equations of motion (indeed in this case, the construction can be realised because of the dimension of the symmetry group). In this way, we could write down a generalised Raychaudhuri equation for each congruence. However, Theorem 7.2 gives us a means of analysing the congruences without explicit integration to obtain expressions for Z.

We illustrate this process with the pair X_1, X_2. In this case using Theorem 7.2 yields

$$[X_b^c] = \begin{pmatrix} -u^1 & 1 \\ -u^2 & 0 \end{pmatrix}$$

and

$$[A_Z] = \begin{pmatrix} 0 & \frac{1}{2}B \\ \frac{1}{2}B & \frac{-BZ^1}{Z^2} \end{pmatrix}$$

so that

$$\theta := \mathrm{tr}(A_Z) = \frac{-BZ^1}{Z^2}.$$

The generalised Raychaudhuri equation is (using Newton's trace formula: see, for example, [3])

$$Z\left(\frac{1}{\theta}\right) = 1 - \frac{1}{\theta^2}\left\{2\sum_{i<j}\lambda_i\lambda_j - \mathrm{tr}(\bar{\Phi})\right\} = 1 - \frac{1}{\theta^2}\{2\det(A_Z) - \mathrm{tr}(\bar{\Phi})\} = 1 + \frac{B^2}{\theta^2}.$$

It is evident that if θ is negative at a point q on this congruence then at some point on the orbit of q, $(1/\theta)$ has a zero, and it can be shown from Eq. (5.3) that the congruence collapses. This occurs at points where $Z^2 = 0$.

It is a simple matter to obtain an explicit representation for Z, either by integrating Eq. (8.1) directly, or using the reduction of order technique of Sherring and Prince [6] (In this case, the fields $\Gamma, X_1, \ldots, X_4$ commute and so the elements of the dual basis, $\{\omega^1, \ldots, \omega^5\}$, are all locally exact. The expression for Z is obtained by locally inverting the expressions $f^4 =$ constant, $f^5 =$ constant for u^1 and u^2 ($\omega^4 = df^4$, $\omega^5 = df^5$).) By either technique we find

$$Z^{1^2} + Z^{2^2} = C^2$$

and

$$Z^1 = Bx^2 + D$$

with $C \in \mathbb{R}^+$, $D \in \mathbb{R}$. We see that $Z^2 = 0$ on the straight lines $x^2 = (-D \pm C)\backslash B$ for a given choice of the two parameters C and D. On these lines the congruence collapses. The congruences in this case are generated by translation of circles in the x^1 direction.

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